Correlation theory of delayed feedback in stochastic systems below Andronov-Hopf bifurcation

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Here we address the effect of large delay on the statistical characteristics of noise-induced oscillations in a nonlinear system below Andronov-Hopf bifurcation. In particular, we introduce a theory of these oscillations that does not involve the eigenmode expansion, and can therefore be used for arbitrary delay time. In particular, we show that the correlation matrix (CM) oscillates on two different time scales: on the scale of the main period of noise-induced oscillations, and on the scale close to the delay time. At large values of the delay time, the CM is shown to decay exponentially only for large values of its argument, while for the arguments comparable with the value of the delay, the CM remains finite disregarding the delay time. The definition of the correlation time of the system with delay is discussed.

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I. INTRODUCTION

We consider a class of systems which do not self-oscillate on their own, but being subject to random fluctuations, are capable of demonstrating nondamped oscillations with pronounced time scales. These oscillations are called noise induced. Among the systems with noise-induced oscillations are the ones just below Andronov-Hopf bifurcation [1], which play an important role in, e.g., neurodynamics [2], condensed matter physics [3], and nonlinear optics [4]. At the values of control parameters close to Andronov-Hopf bifurcation, the fixed point in such systems is a stable focus, i.e., it has a pair of complex-conjugate eigenvalues with nonzero imaginary parts equal to β. Without noise, a small-amplitude relaxation towards the equilibrium is described by damped oscillations with the period given by the inverse of β. However, if additive noise is switched on, energy fluctuations lead to the permanent excitation of the damped oscillations, and as a result, these oscillations are never damped. If noise strength D is small, the power spectrum of such oscillations has a maximum near β, but at large noise this maximum is shifted [5]. The degree of coherence of motion is often described by the correlation time t_{corr}. In general, for some stationary stochastic process $x(t)$, t_{corr} is formally understood as the time interval for which the correlation between $x(s)$ and $x(s+t_{\text{corr}})$, where s is some arbitrary time moment, is negligibly small as compared to the correlation between $x(s)$ and $x(s)$. We will refer to this formal definition of the correlation time as the definition A. t_{corr} is rigorously defined through the autocorrelation function (ACF) $\Psi(t)$ only for processes whose autocorrelation functions decay exponentially, i.e., $\Psi(t)=\exp(-\lambda t)$: then t_{corr} is defined as $1/\lambda$. In an attempt to define the correlation time regardless of the actual shape of the autocorrelation function and of the rate of its decay, the following relation between t_{corr} and the ACF $\Psi(s)$ was introduced in Ref. [6]: $t_{\text{corr}}=1/\Psi(0)\int_0^\infty|\Psi(s)|ds$. This definition is regarded as the definition B. Here we show that at least for a linear stochastic equation with time delay, the definition A is in contradiction with the definition B.

Recently, the possibility of controlling the noise-induced oscillations by means of a noninvasive Pyragas [7] time-delayed feedback was demonstrated on the example of the van der Pol oscillator [5,8–12]. Invasive time-delayed feedback in the deterministic van der Pol oscillator was considered in Ref. [13] and recently in the noisy van der Pol oscillator in Ref. [14]. In the noninvasive case the controlling force $F(t)$ is introduced in the form of the difference between the two values of some state variable x: τ time units before, and at the current time moment t, i.e., $F(t)=K[x(t-\tau)-x(t)]$. It was shown for $K>0$ that the controlling force in this form neither changes the position of the fixed point, nor induces Andronov-Hopf bifurcation in the van der Pol oscillator (see, e.g., [8]). It has also been shown that in the presence of the time-delayed feedback, both the degree of coherence and the frequency of the highest peak in the power spectrum can be altered by changing the delay time τ and the strength of the feedback K.

In the limit of small noise the equations of the system below Andronov-Hopf bifurcation can be linearized around the fixed point, and the power spectrum $S(\omega)$ of the system with delay can be estimated analytically [10]. Alternatively, when noise is not small, one can use mean-field approximation [5,12], which would again result in linear stochastic differential equations with delay and allow for the analytical calculation of $S(\omega)$. On the basis of this estimate, the frequency of the noise-induced oscillations with delayed feedback was shown to have an approximately piecewise-linear dependence on the delay time τ. Despite that the power spectrum $S(\omega)$ is known, the exact expression for the ACF, which is given by the inverse Fourier transform of $S(\omega)$, remains undetermined. An approximate expression for the ACF was used in [5,12,15] in the case when there is a separation of eigenmodes in the spectrum of the characteristic equation. This occurs if the delay time is not too large as compared with the typical time scale of oscillations without the feedback. However, in the limit of large delay times, the spectrum of the eigenmodes is quasicontinuous and the eigenmode expansion cannot be easily truncated.

In this paper we compute the correlation matrix (CM) in the form which does not involve the expansion into an infinite series of the eigenmodes and can therefore be used for arbitrary delay times. The CM is determined as a solution of a linear delay differential equation using a method similar to the one that has already been applied to compute the ACF of a scalar stochastic differential equation [15–17].

With the derived expression for the CM we show that for the values of its argument comparable to τ, the CM remains finite disregarding τ. This means that in the limit of large τ the correlation time grows with τ and does not saturate as $\tau \to \infty$. We show that this fact is in contradiction with the classical relation of the correlation time t_{corr} to the autocorrelation function $\Psi(s)$ via $t_{corr}=1/\Psi(0)\int_0^\infty |\Psi(s)| ds$, since the integral $\int_0^\infty |\Psi(s)| ds$ saturates as $\tau \to \infty$.

II. CORRELATION MATRIX FOR THE NOISY VAN DER POL OSCILLATOR WITH DELAY

Consider the noisy van der Pol oscillator with time-delayed feedback control introduced through the y variable [5,8,10]

$$\dot{x}(t) = y,$$

$$\dot{y}(t) = (\epsilon - x^2)y(t) - \omega_0^2 x(t) + K[y(t-\tau) - y(t)] + D\chi(t), \tag{1}$$

where $K > 0$ is the strength of the controlling force and D is the noise strength. The term χ is the white Gaussian noise with the autocorrelation function $\langle \chi(t)\chi(t')\rangle = \delta(t-t')$. The delayed feedback is taken in the simplest possible form, following the earlier works on the control of noise-induced oscillations [5,8–11].

We consider Eqs. (1) at the values of parameters ϵ and ω_0 when the fixed point $(x=y=0)$ is a stable focus: $0 > \epsilon > -2\omega_0$ and $K > 0$. The choice of the sign of the parameter $\epsilon < 0$ is not in contradiction with the original van der Pol system, as it is shown, for instance, in Ref. [18], where the constant ϵ is given by the sum of two terms: the strength of the feedback in the circuit and the resistance with the negative sign. If the resistance is larger than the feedback term, the overall coefficient ϵ is negative. More recently, the forced van der Pol oscillator with negative (as well as positive) overall damping was used as a model for otoacoustic emission (OAE) in Ref. [19].

The effect of delayed feedback on the dynamics of the van der Pol system can be visible already in realizations shown in Fig. 1 at different values of τ: $\tau=0$, $\tau=50$, and $\tau=100$. One can notice that at $\tau > 0$ the visible modulation of the amplitude of variable $x(t)$ appears, whose average period is approximately equal to τ.

For small noise intensity $D \ll 1$ the amplitude of the noise-induced oscillations is also small, which allows us to linearize Eqs. (1) around the fixed point. However, if the noise strength D is large, the linearization procedure can still be performed using the mean-field approximation [5]. Hereby, the nonlinear term $(\epsilon - x^2)$ in the second equation in Eqs. (1) is replaced by its average value, i.e., by $\epsilon' = \epsilon - \langle x^2\rangle$. The new parameter ϵ' is then determined self-consistently to yield $\epsilon' = \epsilon/2 + \sqrt{(\epsilon/2)^2 + D^2/(2\omega_0^2)}$. The mean-field linearized version of Eqs. (1) is given by

$$\dot{x}(t) = y,$$

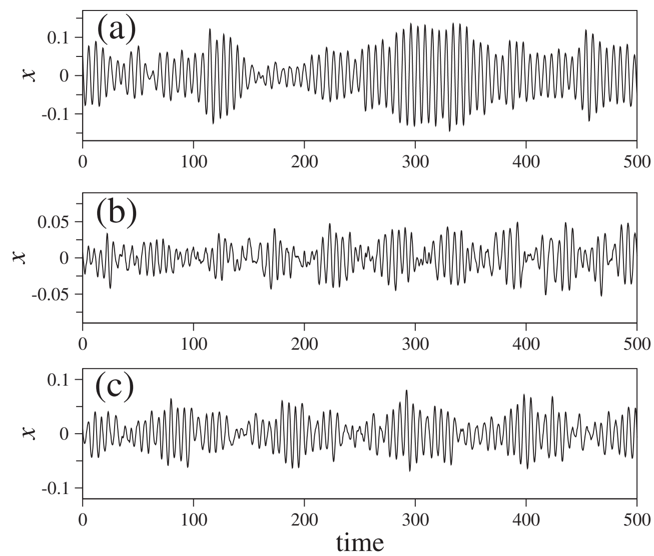

FIG. 1. Realizations of the x variable of the noisy van der Pol oscillator Eqs. (1) at different values of the delay time. (a) $\tau=0$, (b) $\tau=50$, (c) $\tau=100$. Other parameters are $\epsilon=-0.01$, $D=0.01$, $K=0.5$.

$$\dot{y}(t) = \epsilon' y(t) - \omega_0^2 x(t) + K[y(t-\tau) - y(t)] + D\chi(t). \tag{2}$$

In what follows, we use Eqs. (2) to analytically compute the correlation matrix (CM) $\boldsymbol{\Psi}(s)$ of the noisy van der Pol oscillator and compare it with the simulations of the original nonlinear system Eqs. (1).

The general definition of the CM [20] is

$$\boldsymbol{\Psi}(s) = \begin{pmatrix} \Psi_{xx}(s) & \Psi_{yx}(s) \\ \Psi_{xy}(s) & \Psi_{yy}(s) \end{pmatrix}, \tag{3}$$

where its entries are determined as follows:

$$\Psi_{xx}(s) = \langle x_s x\rangle, \quad \Psi_{xy}(s) = \langle x_s y\rangle, \quad \Psi_{yx}(s) = \langle y_s x\rangle, \quad \Psi_{yy}(s) = \langle y_s y\rangle. \tag{4}$$

In Eq. (4), the brackets $\langle \ldots \rangle$ denote the averaging over the ensemble of realizations, and the subscript "s" indicates that the corresponding variable is retarded by s time units. For the convenience of calculations, we make use of the assumption that the random process described by Eqs. (1) is ergodic, i.e., the average of any quantity of our interest over the ensemble of its realizations coincides with the average of the same quantity over time using any single realization of this process of infinitely large duration. The assumption on ergodicity allows one to obtain simpler explicit formulas for the components of CM. For instance, for the second equation in Eqs. (4) we have

$$\Psi_{xy}(s) = \langle x_s y\rangle = \lim_{T\to\infty} \frac{1}{T}\int_0^T x(s'-s)y(s')ds'. \tag{5}$$

From Eqs. (4) the entries of the correlation matrix (3) have the following obvious properties:

$$\Psi_{xy}(s) = \langle x_s y\rangle = \langle y x_s\rangle = \Psi_{yx}(-s),$$

$$\Psi_{xx}(s) = \langle x_s x\rangle = \langle x x_s\rangle = \Psi_{xx}(-s),$$

$$\Psi_{yy}(s) = \langle y_s y \rangle = \langle y y_s \rangle = \Psi_{yy}(-s). \tag{6}$$

This yields the symmetry condition for the correlation matrix [20]

$$\boldsymbol{\Psi}(s) = \boldsymbol{\Psi}^T(-s), \tag{7}$$

where the superscript T denotes the transposed matrix.

By taking the Fourier transform of Eq. (1) and using the Wiener-Khinchine theorem [21], as it was done in [10,11], one finds for the Fourier transforms for Ψ_{xx}, Ψ_{xy}, and Ψ_{yy},

$$\hat{\Psi}_{xy}(\omega)\delta(\omega-\omega') = i\omega\hat{x}^*(\omega)\hat{x}(\omega'),$$

$$\hat{\Psi}_{yx}(\omega)\delta(\omega-\omega') = -i\omega\hat{x}^*(\omega)\hat{x}(\omega'),$$

$$\hat{\Psi}_{yy}(\omega)\delta(\omega-\omega') = \omega^2\hat{x}^*(\omega)\hat{x}(\omega'), \tag{8}$$

where ^ denotes the Fourier transform and * denotes the complex conjugate. From Eq. (8) we conclude that

$$\Psi_{yy}(s) = -\frac{\partial^2\Psi_{xx}(s)}{\partial s^2},$$

$$\Psi_{xy}(s) = \frac{\partial\Psi_{xx}(s)}{\partial s},$$

$$\Psi_{yx}(s) = -\frac{\partial\Psi_{xx}(s)}{\partial s}. \tag{9}$$

Summarizing, we can write the correlation matrix (3) in the form

$$\boldsymbol{\Psi}(s) = \begin{pmatrix} \Phi(s) & -\frac{\partial\Phi(s)}{\partial s} \\ \frac{\partial\Phi(s)}{\partial s} & -\frac{\partial^2\Phi(s)}{\partial s^2} \end{pmatrix}, \tag{10}$$

where $\Phi(s)$ is an even function, which still has to be determined. Note that whereas $\Phi(s)$ and $-\partial^2\Phi(s)/\partial s^2$ are even functions, the off-diagonal entries of the correlation matrix are odd functions of s.

We now determine the function $\Phi(s)$. As it was shown in [22] the CM $\boldsymbol{\Psi}$ given by Eq. (3) obeys the following equation:

$$\dot{\boldsymbol{\Psi}}(s) = \boldsymbol{A}\boldsymbol{\Psi}(s) + \boldsymbol{B}\boldsymbol{\Psi}(s-\tau), \tag{11}$$

with $\boldsymbol{A}$ and $\boldsymbol{B}$ given by

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & (\epsilon' - K) \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} 0 & 0 \\ 0 & K \end{pmatrix}. \tag{12}$$

Following [17] we are looking for the solution of Eq. (11) on the interval $s \in [-\tau, \tau]$ in the form

$$\boldsymbol{\Psi}(s) = \boldsymbol{C}_1 e^{\lambda s} + \boldsymbol{C}_2 e^{\lambda(\tau - s)}, \tag{13}$$

with some unknown matrices $\boldsymbol{C}_1$, $\boldsymbol{C}_2$ and some number λ. From Eqs. (9) we conclude that there are only two unknown constants C and H the matrices $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ depend on, namely,

$$\boldsymbol{C}_1 = C\begin{pmatrix} 1 & -\lambda \\ \lambda & -\lambda^2 \end{pmatrix}, \quad \boldsymbol{C}_2 = H\begin{pmatrix} 1 & \lambda \\ -\lambda & -\lambda^2 \end{pmatrix}. \tag{14}$$

Plugging Eq. (13) into Eq. (11) and using $\boldsymbol{\Psi}(s) = \boldsymbol{\Psi}^T(-s)$ we obtain

$$\lambda\boldsymbol{C}_1 = \boldsymbol{A}\boldsymbol{C}_1 + \boldsymbol{B}\boldsymbol{C}_2^T,$$

$$-\lambda\boldsymbol{C}_2 = \boldsymbol{A}\boldsymbol{C}_2 + \boldsymbol{B}\boldsymbol{C}_1^T. \tag{15}$$

Introducing a relation $H = FC$ with some unknown F, we solve Eq. (15) with respect to λ and F to obtain

$$\lambda = \pm\frac{\sqrt{(\epsilon' - K)^2 - K^2}}{2} \pm i\sqrt{\omega_0^2 - \frac{(\epsilon' - K)^2 - K^2}{4}},$$

$$F_{(\pm)} = -\frac{\epsilon' - K}{K} \pm \frac{\sqrt{(\epsilon' - K)^2 - K^2}}{K}. \tag{16}$$

This suggests the following general form of the real function $\Phi(s)$:

$$\Phi(s) = e^{-\gamma s}[A_1\cos(\Omega s) + A_2\sin(\Omega s)] + F_{(-)}e^{\gamma(\tau - s)} \times [A_1\cos(\Omega(\tau - s)) + A_2\sin(\Omega(\tau - s))], \tag{17}$$

where

$$\gamma = \frac{\sqrt{(\epsilon' - K)^2 - K^2}}{2},$$

$$\Omega = \sqrt{\omega_0^2 - \frac{(\epsilon' - K)^2 - K^2}{4}}. \tag{18}$$

Similar expression for the autocorrelation function has been obtained independently considering harmonic oscillator with damping [23].

The real constants A_1 and A_2 in Eq. (17) must be determined from the normalization conditions for $\Phi(s)$. These additional conditions can be derived from Eqs. (1) as follows. First, rewrite Eqs. (1) in the form of differentials dy, dx, and $d\chi$. Following [21], we compute the average of the differentials $d(x^2) = (dx)^2 + 2xdx$ and $d(y^2) = (dy)^2 + 2ydy$ up to the order of $O(dt)$. Note that $(dy)^2$ is of the order $O(dt)$ due to the relation $d\chi = N(0,1)\sqrt{dt}$, where $N(0,1)$ stands for the normal (Gaussian) distribution with unit variance and zero mean. This yields

$$\frac{\langle d(x(t)^2)\rangle}{dt} = 2\langle x(t)y(t)\rangle,$$

$$\frac{\langle d(y(t)^2)\rangle}{dt} = 2[(\epsilon' - K)\langle y(t)^2\rangle - \omega_0^2\langle x(t)y(t)\rangle + K\langle y(t)y(t - \tau)\rangle] + D^2. \tag{19}$$

When deriving Eqs. (19) we used the fact that $\langle x(t)d\chi(t)\rangle = \langle y(t)d\chi(t)\rangle = 0$ [21]. Now observing that in the stationary limit $\langle d(x(t)^2)\rangle/dt = \langle d(y(t)^2)\rangle/dt = 0$, we obtain

$$0 = \langle x(t)y(t)\rangle,$$

$$0 = (\epsilon' - K)\langle y(t)^2\rangle + K\langle y(t)y(t-\tau)\rangle + \frac{D^2}{2}. \tag{20}$$

Equations (20) yield the normalization conditions for $\Phi(s)$

$$\left.\frac{\partial\Phi(s)}{\partial s}\right|_{s=0} = 0,$$

$$\left.\frac{\partial^2\Phi(s)}{\partial s^2}\right|_{s=0} = -\left(\frac{K}{\epsilon' - K}\right)\left.\frac{\partial^2\Phi(s)}{\partial s^2}\right|_{s=\tau} + \frac{D^2}{2(\epsilon' - K)}. \tag{21}$$

The constants A_1 and A_2 then read

$$A_2 = \frac{D^2}{2N}, \quad A_1 = EA_2, \tag{22}$$

where

$$E = \frac{\Omega + F_{(-)}\exp(-\gamma\tau)[\gamma\sin(\Omega\tau) - \Omega\cos(\Omega\tau)]}{\gamma - F_{(-)}\exp(-\gamma\tau)[\gamma\cos(\Omega\tau) + \Omega\sin(\Omega\tau)]},$$

$$N = E[\epsilon' - K + K(F_{(-)}\ ] + e^{(-\gamma\tau)}[F_{(-)}(\epsilon' - K) + K][E\cos(\Omega\tau) + \sin(\Omega\tau)]. \tag{23}$$

The CM outside the interval $[-\tau, \tau]$ can be obtained from Eq. (11). The recurrent equation for $\Phi_n(s)$ on the interval $[n\tau,(n+1)\tau]$ with $n=1,2,\ldots$ reads

$$\frac{\partial^2\Phi_n(s)}{\partial s^2} = -\omega_0^2\Phi_n(s) + (\epsilon' - K)\frac{\partial\Phi_n(s)}{\partial s} + K\Phi_{n-1}(s), \tag{24}$$

where $\Phi_{n-1}(s)$ is the solution of Eq. (24) on the interval $[(n-1)\tau, n\tau]$.

We compare the derived expression (17) with the CM computed numerically from the original nonlinear system (1). To integrate Eqs. (1) numerically we use the fourth-order Runge-Kutta algorithm adopted for the stochastic delay differential equations. The system parameters are set to $\epsilon = -0.01$, $K=0.5$, $\omega_0=1$, $D=0.1$. For every fixed value of the delay time τ, Eqs. (1) were integrated over the time domain larger than at least 300τ and then averaged over 1000 realizations. The results were verified by varying the time steps in the range between 10^{-3} and 10^{-4}.

Figure 2 shows the autocorrelation functions at $\tau=10$. Solid lines in each panel show the corresponding function from Eq. (10) with $\Phi(t)$ given by Eq. (17); circles were obtained from simulation. As we can see, the mean-field approximation gives an adequate estimate for the CM at the values of the noise strength D of up to 10^{-1}.

The knowledge of the correlation matrix allows us to compute the variances $\langle x^2\rangle$ and $\langle y^2\rangle$ as functions of the delay time τ. From Eq. (10) it follows that $\langle y^2\rangle = \omega_0^2\langle x^2\rangle - K\langle y_\tau x\rangle$. However, since the absolute value of $K\langle y_\tau x\rangle$ is small as compared to $\langle x^2\rangle$ and $\omega_0=1$, the difference between $\langle y^2\rangle$ and $\langle x^2\rangle$ is marginal. Therefore, we plot only $\langle x^2\rangle$ as a function of the delay time τ. This is shown in Fig. 3. The mean-field analytic expression in units of D^2 (solid line) is compared with the

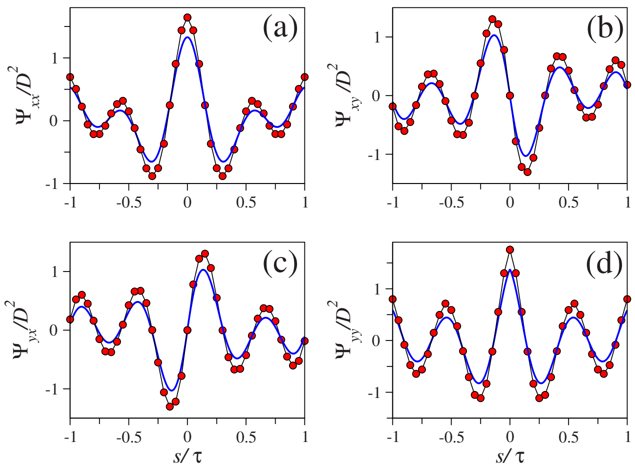

FIG. 2. (Color online) Autocorrelation and cross-correlation functions in the units of D^2 (a) Ψ_{xx}, (b) Ψ_{xy}, (c) Ψ_{yx}, and (d) Ψ_{yy} for $\tau=10$. Solid lines correspond to the analytic expression (10) with $\Phi(s)$ given by Eq. (17); symbols show the corresponding functions calculated numerically.

variance calculated numerically (circles). Figure 3(a) represents thecomparison for small noise strength $D=0.003$; Fig. 3(b) shows the variance computed at large noise strength $D=0.1$. As we see, the variance is maximal at $\tau=0$. As the delay time increases, $\langle x^2\rangle$ oscillates with decreasing amplitude. These oscillations occur on the time scale of $2\pi/\Omega$ and their amplitude decays with γ as it can be easily extracted from Eqs. (17) and (23). As $\tau\to\infty$ the variance tends asymptotically to $\gamma D^2/[2\Omega(\epsilon' - K + KF_{(-)})]$.

III. LARGE DELAY TIMES

We focus on the behavior of the correlation matrix in the limit of large delay times. As we mentioned above, $S_x(\omega)$ can be computed through the Fourier transforms of the linearized van der Pol system Eqs. (2) following [10] to yield

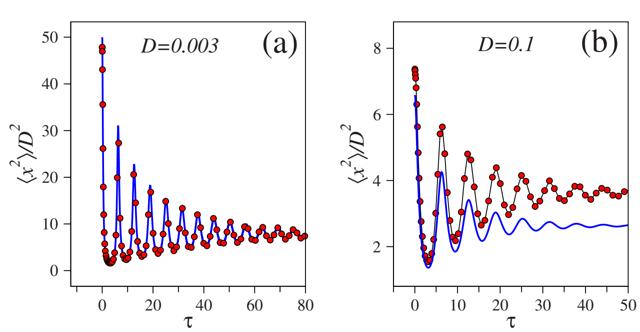

FIG. 3. (Color online) Variance $\langle x^2\rangle = \Psi_{xx}(0)$ in the units of D^2 vs delay time. Solid line corresponds to the analytic expression (17); symbols show the result of simulation. (a) $D=0.003$, (b) $D=0.1$.

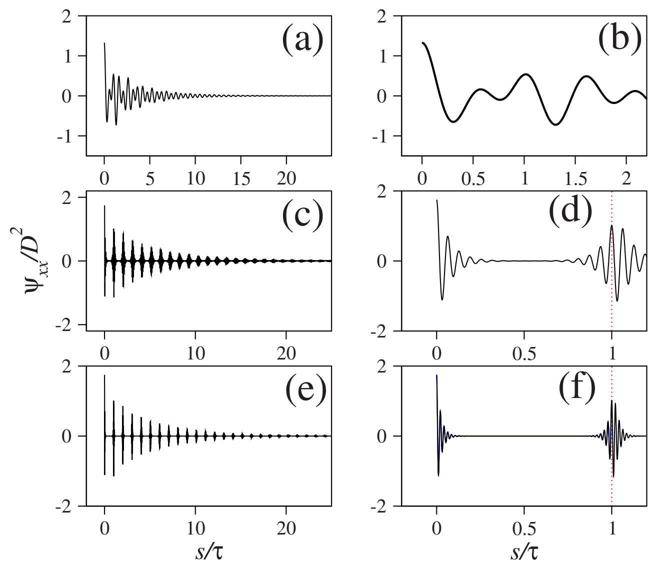

FIG. 4. (Color online) Analytic autocorrelation function Ψ_{xx} in the units of D^2 at different values of the delay time τ: [(a) and (b)] $\tau=10$, [(c) and (d)] $\tau=100$, [(e) and (f)] $\tau=300$. Panels (a), (c), and (e) show the behavior of the CM on the scale of $s=25\tau$. Panels (b), (d), and (f) reveal the behavior on the scale of $s\approx\tau$.

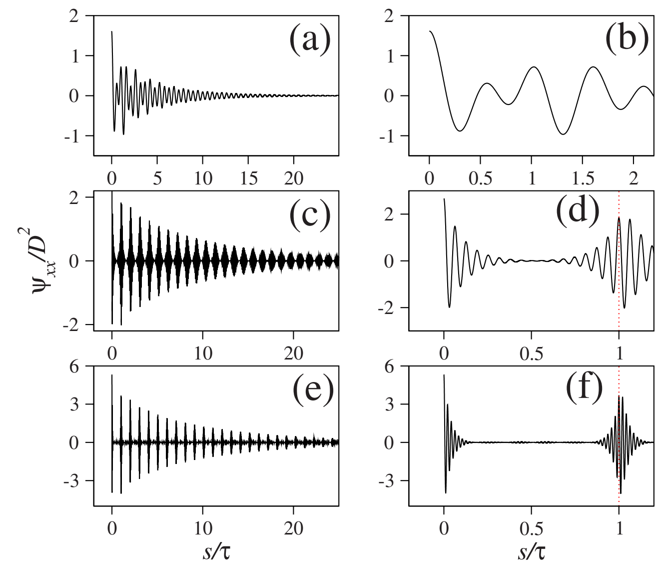

FIG. 5. (Color online) Numerical autocorrelation function Ψ_{xx} in the units of D^2 at different values of the delay time τ: [(a) and (b)] $\tau=10$, [(c) and (d)] $\tau=100$, [(e) and (f)] $\tau=300$. Panels (a), (c), and (e) show the behavior of the CM on the scale of $s=25\tau$. Panels (b), (d), and (f) reveal the behavior on the scale of $s\approx\tau$.

$$S_x(\omega) = \frac{D^2}{[\omega^2-\omega_0^2+K\omega\sin(\omega\tau)]^2+\omega^2[\epsilon'+K((\cos(\omega\tau)\ -1)]^2}. \tag{25}$$

Figures 4(a), 4(c), and 4(e) show the mean field $\Phi(s)$ obtained by calculating the inverse Fourier transform of Eq. (25) numerically for the parameter values used for Fig. 2 and the delay times $\tau=10$, $\tau=100$, and $\tau=300$, respectively. Note that on the interval $s\in[-\tau,\tau]$ the autocorrelation function computed by numerically taking inverse Fourier transform of Eq. (25) coincides with high accuracy with Eq. (17) (not shown). Figures 4(b), 4(d), and 4(f) show $\Phi(s)$ on the intervals $[0,\tau]$ and $[0,2\tau]$. For the purpose of comparison we also plot $\Phi(s)$ computed numerically by solving the nonlinear system (1). This is done in Fig. 5, where the parameters for all the panels are the same as in the corresponding panels in Fig. 4. As we see, with growing delay time the shape of the autocorrelation function becomes similar to a fish bone. It has two different periodic components: the first one is given by $2\pi/\Omega$, and the second approximately equals the delay time τ.

A closer look at the autocorrelation function reveals, however, that at large τ its second period is somewhat different from τ. From Figs. 4(d) and 4(f) and Figs. 5(d) and 5(f), as well as from the analytic expression (17), it can be inferred that at small values of the argument s the amplitude of $\Phi(s)$ decays exponentially with the exponent given by γ from Eq. (18). At $s>\tau/2$ the amplitude grows exponentially with the same exponent γ and reaches the value of approximately $\Phi(0)F_{(-)}$ at $s=\tau$. This point is highlighted in Figs. 4(d) and 4(f) and Figs. 5(d) and 5(f) by the dotted vertical lines. However, for $s>\tau$ the amplitude of the CM keeps on growing until it achieves its second maximum at approximately $s=\tau+2\pi/\Omega$. The third maximum of the amplitude is reached at $s=\tau+4\pi/\Omega$, and so on.

Interestingly, as τ increases, the autocorrelation function at $s=\tau$ remains finite; i.e. it does not tend to zero. In fact $\Phi(\tau)\to F_{(-)}\Phi(0)$ as $\tau\to\infty$. This is a direct consequence of the presence of a delay term in Eqs. (1). Namely, by introducing the delay term into the system (1) we have introduced the correlation between the values of the stochastic process taken at times separated by τ time units from one another. This correlation persists for all delay times, whatever large.

A. Frequency shift due to delay

The main frequency of the noise-induced oscillations can be defined as the frequency that corresponds to the global maximum (highest peak) of the power spectrum [8]. However, when considering a system of two stochastic equations, such as Eqs. (1), one can compute two different power spectra. The first spectrum S_x can be calculated from the variable $x(t)$, and the second S_y can be computed from the realizations $y(t)$. Therefore, S_x is given by the Fourier transform of $\Psi_{xx}(s)$ and S_y by that of $\Psi_{yy}(s)$. From Eqs. (9) we recall that $S_y=\omega^2 S_x$. Interestingly, S_x and S_y have different main frequencies. For instance, at $\tau=0$ these frequencies are given by

$$(\omega_m)_x = \sqrt{\omega_0^2-\frac{(\epsilon')^2}{2}},$$

$$(\omega_m)_y = \omega_0, \tag{26}$$

where $(\omega_m)_x$ is the main frequency of $S_x(\omega)$ and $(\omega_m)_y$ is the main frequency of $\omega^2 S_x(\omega)$. There is yet another frequency which is different from both $(\omega_m)_x$ and $(\omega_m)_y$. This is given by the imaginary part $\mathrm{Im}(\lambda_m)$ of the least stable eigenvalue of system Eqs. (1). At $\tau=0$ it has the value of $\mathrm{Im}(\lambda_m) = \sqrt{\omega_0^2-(\epsilon')^2/4}$. Only for $\epsilon' \ll \omega_0$ all three frequencies coincide.

As it was shown in [8,10], the frequency $(\omega_m)_y$ depends almost piecewise linearly on τ. It increases almost linearly with τ on some interval of τ, then it drops discontinuously, and then increases almost linearly again. From Eq. (25) it is clear that for finite τ, the frequency $(\omega_m)_x$ shows qualitatively similar dependence on τ.

However, in the limit of *large* delay times, the behavior of $(\omega_m)_y$ and $(\omega_m)_x$ becomes essentially different. Using Eq. (25) one can show that in the limit $\tau\to\infty$ the frequency $(\omega_m)_x$ tends to its limiting value, which is different from the corresponding frequency at $\tau=0$, whereas the limiting value of $(\omega_m)_y$ coincides with its value at $\tau=0$.

To show this, we notice that at large τ the power spectrum S_x oscillates in the frequency space with the period of approximately $2\pi/\tau$. Using this fact, we introduce the "background function" $S_1(\omega)$ as the limit of the running average of the expression Eq. (25) over $2\pi/\tau$,

$$S_1(\omega) = \lim_{\tau\to\infty}\frac{\tau}{2\pi}\int_{\omega}^{\omega+2\pi/\tau} S_x(\omega')d\omega'. \qquad (27)$$

This yields

$$S_1(\omega) = \frac{D^2}{[\omega^2-\omega_0^2]^2+\omega^2[(\epsilon')^2-2\epsilon' K]}. \qquad (28)$$

The background function S_2 for $S_y(\omega)=\omega^2 S_x(\omega)$ is obtained by multiplying S_1 by ω^2. The background function at the given frequency ω shows the value around which the spectrum oscillates, and is a convenient way to characterize the shape of the spectrum *without* its oscillating component with period $2\pi/\tau$ over frequency ω. In the limit as $\tau\to\infty$ the main frequency of the power spectrum S_x tends to $(\omega_m)_x = \sqrt{\omega_0^2-[(\epsilon'-K)^2-K^2]/2}$, which is different from the frequency without the feedback given by $\sqrt{\omega_0^2-(\epsilon')^2/2}$. The limiting value of the main frequency of S_y as $\tau\to\infty$ is given by ω_0, i.e., it remains unshifted with respect to its value at $\tau=0$. For $(\omega_m)_x$ the shift of the frequency of noise-induced oscillations is

$$\Delta\omega_m = \sqrt{\omega_0^2-\frac{(\epsilon'-K)^2-K^2}{2}} - \sqrt{\omega_0^2-\frac{\epsilon'}{2}} < 0. \qquad (29)$$

Figure 6(a) shows the mean-field power spectrum (25) for $\epsilon=-0.5$, $\omega_0=1$, $K=0.5$, $\tau=0$ (solid line), and $\tau=100$ (dotted line). The respective background spectrum (28) is shown by the dashed line. Numerical spectra together with the background spectrum are shown in Fig. 6(b). It can easily be seen from Figs. 6(a) and 6(b) that as τ increases, the frequency of the noise-induced oscillations is shifted towards smaller values, as compared to their frequency without the feedback control.

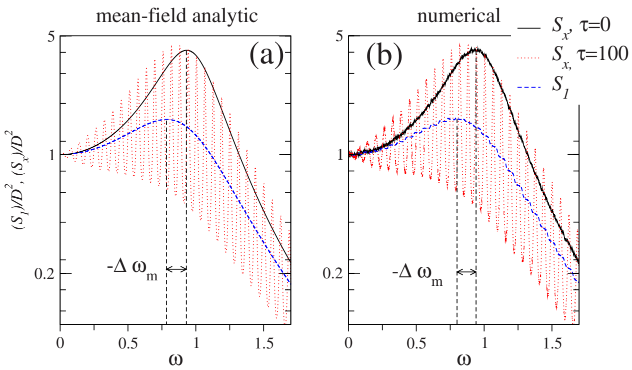

FIG. 6. (Color online) (a) Analytic mean-field power spectrum $S_x(\omega)$ Eq. (25) for $\tau=0$ (solid line), $\tau=100$ (dotted line), and the background spectrum $S_1(\omega)$ (dashed line). Other parameters are $-\epsilon=K=0.5$, $\omega_0=1$. The frequency difference $\Delta\omega_m$ is given by Eq. (28). (b) Numerical power spectrum and numerical background function at the same parameters as in (a).

B. Correlation time of a process with delay

The correlation time t_{corr} of some stochastic process $x(t)$ with the autocorrelation function (ACF) $\Psi_x(s)$ can be qualitatively understood as the time interval δs, such that the correlation between $x(t)$ and $x(t\pm\delta s)$ is negligibly small; i.e., $\Psi_x(s>\delta s) \ll \Psi_x(0)$. If the ACF decays as $\sim\exp(-\gamma t)$, the correlation time is given by the inverse of the order of this exponent, i.e., $t_{\mathrm{corr}}=1/\gamma$. But how should one determine t_{corr} if the behavior of the ACF is nonexponential on some interval of its argument? The (arguably) most popular formula for t_{corr} disregarding the actual shape of the ACF reads [6]

$$t_{\mathrm{corr}} = \int_0^{\infty} |\bar{\Psi}_x(s)| ds, \qquad (30)$$

where $\bar{\Psi}_x(s)$ is the ACF $\Psi_x(s)$ divided by $\Psi_x(0)$, so that $\bar{\Psi}_x(0)=1$. This expression is expected to provide at least a qualitatively correct dependence of t_{corr} on the system parameters.

However, here we show that Eq. (30), in fact, cannot be used as the definition of the correlation time in the stochastic equations with time-delayed feedback in the limit of large delay times. The reason is that although the integral in Eq. (30) does saturate (tends to a finite limit) at $\tau\to\infty$, and thus gives some finite value for the proposed t_{corr}, this value does not have the meaning of correlation time in the sense of the first paragraph of this subsection. The reason is that ACF remains finite at the values of its argument comparable with τ, and therefore for infinitely large τ, the correlation time, which should be bigger than τ, appears infinitely large, too. Therefore, for infinitely large delay τ, Eq. (30) gives a finite value of t_{corr}, which contradicts its physical meaning that follows from the autocorrelation function. The easiest way to show this rigorously is to consider a scalar stochastic equation with time delay, that has been considered previously by other authors, e.g., [15,16]

$$\frac{dx(t)}{dt} = ax(t) + bx(t-\tau) + D\chi(t), \tag{31}$$

where $a<0$ and $-a>b>0$ to ensure that the fixed point $x=0$ is stable and that the ACF Ψ_x is strictly positive [16]. The latter obeys the equation

$$\frac{d\Psi_x(s)}{ds} = a\Psi_x(s) + b\Psi_x(s-\tau). \tag{32}$$

This equation was solved in [16,17] and the power spectrum $S_x(\omega)$ was calculated in [17] as a Fourier transform of $\Psi_x(s)$. Since for this process $\Psi_x(s)>0$, the integral $\int_0^\infty|\bar{\Psi}_x(s)|\,ds$ is given by half of the Fourier transform $S_x(\omega)$ of $\bar{\Psi}_x(s)$ taken at zero frequency, $\omega=0$.

$$S_x(0) = \frac{1}{p_1}\left[1 + \frac{(p_2-p_1)}{\sqrt{p_1}}\frac{1-e^{(-2\sqrt{p_1p_2}\tau)}}{\sqrt{p_1}+\sqrt{p_2}+(\sqrt{p_2}-\sqrt{p_1})e^{(-2\sqrt{p_1p_2}\tau)}}\right], \tag{33}$$

where $p_1=-(a+b)/2$ and $p_2=(b-a)/2$.

As we see from Eq. (33), as $\tau\to\infty$, $S(0)$ remains finite, and so is the integral $\int_0^\infty|\psi_x(s)|\,ds$ that would define t_{corr} according to Eq. (30). However, the autocorrelation function at $s=\tau$, when τ is large, is given by [17]

$$\Psi_x(\tau) = \Psi_x(0)\frac{\sqrt{p_2}-\sqrt{p_1}}{\sqrt{p_2}+\sqrt{p_1}}, \tag{34}$$

and therefore it remains finite in the limit $\tau\to\infty$. The correlation time, which must be at least larger than τ, should tend to infinity here, which is in contradiction with the finite value that results from Eq. (33).

A similar conclusion can be made with respect to the CM of a system of stochastic equations such as Eqs. (1). The finiteness of the CM at $s=\tau$ was already shown in Sec. II. In order to show that the integral in Eq. (30) saturates as $\tau\to\infty$, we calculate it with Ψ_x given by the ACF of x, i.e., with $\Psi_x=\Psi_{xx}$. To do so, we numerically calculate inverse Fourier transform of the power spectrum (25) and then compute the integral of its absolute value. The result is shown for different values of the noise strength D in Figs. 7(a) and 7(b) by the dashed line. One can see that the integral in Eq. (30) oscillates on the scale close to $2\pi/\Omega$, where Ω is given by Eq. (18), with decreasing amplitude and saturating background.

Consequently, instead of the integral in Eq. (30), a different quantity has to be used as the definition of the correlation time. One such quantity that has a correct limiting behavior, i.e., that is divergent in the limit of large delay times is the inverse of the real part of the least stable eigenvalue λ_m of Eqs. (32) or (1). As it was shown in [16], the asymptotic behavior of the autocorrelation function at large values of its argument is close to an exponential decay with the exponent given by $1/|\text{Re}(\lambda_m)|$. In [10] the quantity $2/(\pi|\text{Re}(\lambda_m)|)$ was used to estimate the correlation time of the stochastic process Eq. (1) at moderate values of the delay time. Here we show the dependence of $2/(\pi|\text{Re}(\lambda_m)|)$ on τ for large delay times (solid line in Fig. 7) and compare it with the integral in Eq. (30). Clearly, $2/(\pi|\text{Re}(\lambda_m)|)$ is a better approximation for the correlation time, because it diverges in the limit $\tau\to\infty$. The mismatch between $2/(\pi|\text{Re}(\lambda_m)|)$ and t_{corr} from Eq. (30) becomes even more pronounced for large noise strength D, as it is shown in Fig. 7(b).

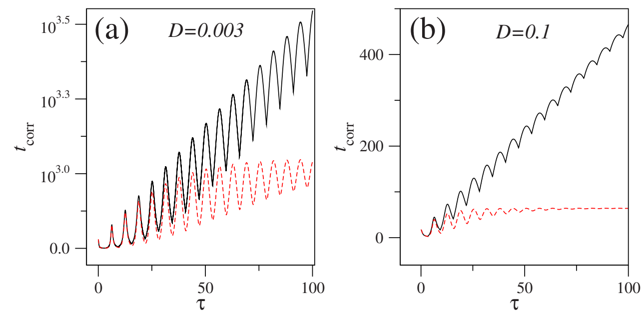

FIG. 7. (Color online) (a) Inverse of the modulus of the real part of the least stable eigenvalue of Eqs. (1) multiplied by $2/\pi$ (solid line) and the integral of the normalized mean-field ACF Ψ_{xx} Eq. (33) (dashed line) vs delay time. Parameters are $D=0.003$, $\epsilon=-0.01$, $K=0.5$. (b) The same as in (a) for larger noise strength $D=0.1$.

IV. CONCLUSION

To conclude, we have derived analytic expression for the correlation matrix (CM) of the noisy van der Pol oscillator below the supercritical Andronov-Hopf bifurcation with time-delayed feedback, which was linearized in the mean-field approximation as in [10]. At small values of its argument s, the derived CM Eq. (17) decays exponentially with the order γ from Eq. (18), which does not depend on τ. The CM oscillates with two different periods: the first period is given by $2\pi/\Omega$, and the second is close to the delay time τ. At large τ, the autocorrelation function decays exponentially with the order of exponent γ for $s<\tau/2$, and grows with the same exponent to the order γ for $\tau>s>\tau/2$. The CM resembles a fish bone with the intervals where it decays and intervals where it grows again. The length of these intervals equals approximately $\tau/2$.

In the limit of large delay times, we have shown that the frequency of the noise-induced oscillations, defined as the frequency of the highest peak in the power spectrum computed from realization $x(t)$, shifts towards smaller frequencies (larger periods) as compared to the frequency without the feedback.

Finally, we demonstrated that the (arguably) most popular definition of the correlation time through the integral of the normalized ACF fails to provide an adequate estimate for the length of the interval of time, when all the correlations disappear, in the case of the linear delay stochastic differential equations. This is due to the fact that the integral Eq. (30) saturates as $\tau\to\infty$, whereas the autocorrelation function remains finite at $s=\tau$, ensuring that the length of the interval of time, when all the correlations disappear, diverges in the limit of large delays. Instead, for linear stochastic differential equations, the correlation time can be estimated as an inverse of the modulus of the real part of the least stable eigenvalue of the characteristic equation.

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